M.A./M.Sc. (Mathematics) Semester-III DR. SU[illegible] S.B. Coll[illegible]

Paper-XI (Functional Analysis)

Unit 1:- Normed linear Space

Example:- An example of a normed linear space (n.l.s.) which is not a Banach space.

Let E be the linear space of all polynomial defined over $[0,1]$ with real co-efficients. For any polynomial $p \in E$, we define,

$$\|p\| = \sup_{t \in [0,1]} |p(t)|.$$

Clearly, $\|p\| \geq 0$. Also $\|p\| = 0$ iff $p = 0$

Again, $\|\alpha p\| = \sup_{t \in [0,1]} |\alpha p(t)|$.

$$= \sup_{t \in [0,1]} |\alpha . p(t)| = |\alpha| . \sup_{t \in [0,1]} |p(t)|.$$

$$= |\alpha| . \|p\|.$$

Now, for any $p, q \in E$.

$$\|p+q\| = \sup_{t \in [0,1]} |(p+q)(t)|$$

$$= \sup_{t \in [0,1]} |p(t) + q(t)|$$

$$\leq \sup_{t \in [0,1]} [|p(t)| + |q(t)|]$$

$$= \sup_{t \in [0,1]} |p(t)| + \sup_{t \in [0,1]} |q(t)|$$

$$= \|p\| + \|q\|$$

So, $\|p+q\| \leq \|p\| + \|q\|$.

$\therefore$ E is a normed linear space.

The metric d generated by the norm on E is given by,

$$d(p,q) = \|p-q\| = \sup_{t \in [0,1]} |p(t) - q(t)|.$$

We shall show that (E, d) is not a complete variable space.

We consider a sequence $\{p_n\}$ of polynomials in E, where $p_n(t) = 1 + t + \frac{t^2}{\lfloor 2} + \cdots + \frac{t^n}{\lfloor n}$.

Let $\epsilon > 0$ be given, then $d(p_n, p_m)$.

$$= \| p_n - p_m \|.$$

$$= \sup_{t \in [0,1]} | p_n(t) - p_m(t) |$$

$$= \sup_{t \in [0,1]} \left| \frac{t^{m+1}}{\lfloor m+1} + \cdots + \frac{t^n}{\lfloor n} \right| \cancel{\leq \frac{1}{\lfloor m+1} + \cdots}$$

$$\leq \frac{1}{\lfloor m+1} + \cdots + \frac{1}{\lfloor n} \to 0$$

as, $m, n \to \infty$.

So, $\{p_n\}$ is a Cauchy sequence in E.

But $\lim_{n \to \infty} p_n(t)\ 1 + t + \frac{t^2}{\lfloor 2} + \cdots$

$= e^t$.

So, $\{p_n\}(t)\}$ converges to e^t, which is not a polynomial in E. So $e^t \notin E$. Since there exist a Cauchy sequence in E, which does not converges in E,

So, E is not a Banach space.

Example:- An example of metric space which is not a normed linear space.

Let S be the linear space of all numerical sequences, for $x = \{x_i\}$ & $y = \{y_i\}$

(61)

in S, we define $x+y=\{x_i+y_i\}$ & $\lambda x=\{\lambda x_i\}$.
We Consider a metric ρ on S defined by,

$$\rho(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}$$

Then, (S,ρ) is a metric space.
We shall show that it is not possible to define a normed on S which generate the metric ρ. For let there be a norm on S and let d be the metric generated by this norm.
It is sufficient to show that d is not equivalent to ρ.

For this we consider a sequence. $e^{(i)}=\{0,0,---1,0,0---\}\in S$.

We define $x^{(i)}=\dfrac{e^i}{\|e^i\|}$, $i=1,2,3,----$

Then, $\rho(x^{(i)},0)\leq\frac{1}{2^i}\to 0$ as $i\to\infty$

So, $x^i \xrightarrow{\rho} 0$

But $d(x^{(i)},0)=\|x^{(i)}-0\|=\|x^{(i)}\|=1$

So, $x^i \not\xrightarrow{d} 0$

Then, d is not equivalent to ρ.
So, S is not a normed linear space.

Theorem QNo→ Let E be a normed linear space. Then for every $x, y \in E$,

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\|.$$

Or, QNo→ In a normed linear space E,

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\| \text{ for all } x, y \in E.$$

Proof:- We have $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$

$$\therefore \|x\| - \|y\| \leq \|x - y\| \quad \text{--- (1)}$$

Again, $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\|$

or, $\|y\| \leq \|x - y\| + \|x\|$

i.e. $\|y\| - \|x\| \leq \|x - y\|$

i.e. $-(\|x\| - \|y\|) \leq \|x - y\| \quad \text{--- (2)}$

From (1) & (2), we have

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x - y\|.$$

Theorem QNo→ If a n.l.s. E, the norm function $\|\cdot\| : E \to \mathbb{R}$ is Continuous.

Proof:- Let $x \in E$ be arbitrary, and let $\{x_n\}$ be a sequence in E Converging to x. Then $\|x_n - x\| \to 0$ as $n \to \infty$, Now, $\big|\,\|x_n\| - \|x\|\,\big| \leq \|x_n - x\| \to 0$ as $n \to \infty$

$\therefore \|x_n\| \to \|x\|$ as $n \to \infty$.

So norm is a Continuous function.

Theorem QNo→ Let E be a n.l.s over a field K. Then the map $(x, y) \to x + y$ of $E \times E \to E$ and $(\alpha, x) \to \alpha x$

of $K \times E$ are Continuous.

Proof:- Let $(x, y) \in E \times E$ be arbitrary. Also let $\{x_n\}$ & $\{y_n\}$ be two sequences in E converging to x & y respectively.

Then, $\|x_n - x\| \to 0$ & $\|y_n - y\| \to 0$ as $n \to \infty$

Now, $\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\|$

$\leq \|x_n - x\| + \|y_n + y\| \to 0$ as $n \to \infty$.

$\therefore x_n + y_n \to x + y$ as $n \to \infty$. Hence the map $(x, y) \to x + y$ is Continuous. Again, let $\{\alpha_n\}$ be a sequence in K converging to α. Then $|\alpha_n - \alpha| \to 0$ as $n \to \infty$.

Then, $\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$

$= \|\alpha_n (x_n - x)\| + \|x(\alpha_n - \alpha)\|$

$\leq \|\alpha_n (x_n - x)\| + \|x(\alpha_n - \alpha)\|$

$= |\alpha_n| \cdot \|x_n - x\| + |\alpha_n - \alpha| \, \|x\|$

$\to |\alpha_n| \cdot 0 + 0 \cdot \|x\| = 0$ as $n \to \infty$

$\therefore \alpha_n x_n \to \alpha x$ So, the map $(\alpha, x) \to \alpha x$ is Continuous.

i.e. the scalar multiplication is a Continuous map.

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QNo $\to$ Show that a non-zero normed linear space E is a Banach space iff $\{x : \|x\| = 1\}$ is Complete.

or, QNo $\to$ A non-zero nls E is a Banach space iff the set $\{x \in E : \|x\| = 1\}$ is Complete.

Proof: - Let $S = \{x \in E : \|x\| = 1\}$ then $S \subseteq E$.
Now, let E be a Banach space. Let $\{x_n\}$ be any Cauchey sequence in S. Then $\{x_n\}$ is a Cauchy sequence in E. Since E is a Banach space, there exists $x \in E$ such that $x_n \to x$ as $n \to \infty$. Since norm function is continuous, $\|x_n\| \to \|x\|$

i.e. $\lim_{n \to \infty} \|x_n\| = \|x\|$ Now, $\|x_n\| = 1$ for all n.

$\therefore \lim_{n \to \infty} \|x_n\| = 1$, i.e. $\|x\| = 1$ i.e. $x \in S$

As every Cauchy sequence is convergese to S therefore, S is complete.
Conversely, let S be complete. Let $\{y_n\}$ be any Cauchy sequence in E. Let $x_n = \dfrac{y_n}{\|y_n\|}$ for every n.

Then, $\|x_n\| = \left\| \dfrac{y_n}{\|y_n\|} \right\| = 1$ for every n.

$\therefore x_n \in S$ for every n. i.e. $\{x_n\}$ is a sequence in S.

We shall show that $\{x_n\}$ is a Cauchy sequence in S, we have

$$\|x_m - x_n\| = \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$= \left\| \frac{y_m}{\|y_m\|} + \frac{y_n}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$= \left\| \frac{y_m - y_n}{\|y_m\|} + \frac{y_n(\|y_n\| - \|y_m\|)}{\|y_m\| \cdot \|y_n\|} \right\|$$

$$\leq \frac{\|y_m - y_n\|}{\|y_m\|} + \frac{\left|\|y_m\| - \|y_n\|\right|}{\|y_m\|}$$

i.e. $\|x_m - x_n\| \leq \frac{2\|y_m - y_n\|}{\|y_m\|} \to 0$ as $m, n \to \infty$

[$\because \{y_m\}$ is bounded $\therefore \|y_m\|$ is finite]

$\therefore \{x_n\}$ is a Cauchy sequence in S, Since S is Complete $x_n \to x \in S$.

Now, $x_n = \frac{y_n}{\|y_n\|}$ $\therefore y_n = \|y_n\| \cdot x_n$

$\therefore \lim_{n\to\infty} y_n = \lim_{n\to\infty} \|y_n\| \cdot x = \alpha x$ (Suppose)

Also, $\alpha x \in E$ $\therefore y_n \to \alpha x \in E$.

So every Cauchy sequence in E Converges in E.

Hence, E is Banach Space.